Discrete Mathematics 62 (1986) 65–70
North-Holland

UNIQUELY COLOURABLE m-DICHROMATIC ORIENTED GRAPHS*

V. NEUMANN-LARA
Instituto de Matemáticas, Universidad Nacional Autónoma de México, México, 04510 D.F., México

N. SANTORO
School of Computer Science, Carleton University, Ottawa, Canada

J. URRUTIA
Computer Science Department, University of Ottawa, Ottawa, Canada

Received 24 August 1984
Revised 8 October 1985

The dichromatic number $d_k(D)$ of a digraph D is the minimum number of colours needed to colour $V(D)$ in such a way that no monochromatic directed cycle is obtained. A digraph D is called *uniquely colourable* if any acyclic $d_k(D)$-colouring of $V(D)$ induces the same partition of $V(D)$. In this paper we construct an infinite family of uniquely colourable m-dichromatic oriented graphs for all $m \geq 2$.

1. Introduction and terminology

An acyclic m-colouring of a digraph D is a colouring of the vertices of D with m colours in such a way that no monochromatic directed cycle is obtained. The *dichromatic number* $d_k(D)$ of D is the minimum number m such that there exists an acyclic m-colouring of D. The dichromatic number was introduced independently by Neumann–Lara [5] and Jacob and Meyniel [4] and has been studied in several papers; see [1–7].

A digraph D is called *uniquely colourable* if every acyclic $d_k(D)$-colouring of D induces the same partition of $V(D)$. Clearly a uniquely colourable m-dichromatic digraph G^* can be obtained from a uniquely colourable m-chromatic graph G by substituting each edge $[u, v]$ of G by the arcs (u, v) and (v, u). It is thus natural to ask for the existence of uniquely colourable m-dichromatic oriented graphs. In this paper we study uniquely colourable oriented graphs and obtain an infinite family of uniquely colourable m-dichromatic oriented graphs for every $m \geq 2$. Some techniques introduced in [7] to study vertex critical m-dichromatic tournaments are used in this paper.

In Section 2, we construct two families of uniquely colourable 2-dichromatic

* This work was supported in part by the Natural Sciences and Engineering Research Council of Canada under grant numbers A2415 and A0977.

oriented graphs. These families are then used in Section 3 to generate uniquely colourable r-dichromatic oriented graphs for $r \geqslant 3$.

Let D be a digraph, $V(D)$ and $A(D)$ will denote the vertex and arc sets of D respectively.

For a vertex $v \in V(D)$, $d^+(v)$, $d^-(v)$, $\Gamma^+(v)$ and $\Gamma^-(v)$ will denote the out and in-degree of v and the out and in-neighborhood of v respectively. The set $\{0, 1, 2, \ldots, n-1\}$ will be denoted by I_n.

Given a subset $S \subset V(D)$, $D[S]$ will denote the subdigraph of D induced by S. In this paper the word cycle will refer only to directed cycles. All digraphs considered here are oriented graphs.

2. Constructing uniquely colourable 2-dichromatic oriented graphs

In this section we will obtain a family of uniquely colourable 2-dichromatic oriented graphs. The following definitions are needed:

For $i_1, i_2, \ldots, i_s \in I_n - \{0\}$, let $\vec{C}_n(i_1, i_2, \ldots, i_s)$ be the digraph with vertex set I_n whose arcs are the ordered pairs $(j, j+i_r)$; $j \in I_n$, $1 \leqslant r \leqslant s$, where $j + i_r$ is taken mod n.

Let $H_r = \vec{C}_{2r+1}(1, 2, \ldots, r-1, r+1)$. Observe that H_r is a regular tournament with $2r+1$ vertices. In [7] it was proved (see Theorem 2) that $d_k(H_r) = 3$.

Let i be a vertex of H_r, $r \geqslant 4$. The following properties can be proved.

(i) $d^+(i) = d^-(i) = r$.

(ii) The subtournaments induced in H_r by $\Gamma^+(i)$ and $\Gamma^-(i)$ respectively contain directed cycles.

(iii) Let T_r be an acyclic subtournament of H_r with r vertices. If the source of T_r is vertex i, then

$$V(T_r) = \{i, i+1, \ldots, i+r-1\} = S_i, \text{ or}$$
$$V(T_r) = S_i \cup \{i+r+1\} - \{i+1\} = S_i',$$

(iv) In $H_r[V(H_r) - S_i']$, $r \geqslant 4$, the in-degree $d^-(v)$ of any vertex is at least 2.

We can now prove the following result.

Theorem 1. *$H_r' = H_r - 0$ is uniquely 2-colourable ($r \geqslant 4$) with chromatic classes S_1 and S_{r+1}.*

Proof. The sets $S_1 = \{1, 2, \ldots, r\}$ and $S_{r+1} = \{r+1, \ldots, 2r\}$ induce acyclic subtournament of H_r', and since H_r' is not acyclic, then $d_k(H_r') = 2$. We shall now prove that H_r' is also uniquely 2-colourable. Let C_0 and C_1 be the chromatic classes of an acyclic 2-colouring of H_r'. Then $H_r'[C_0]$ and $H_r'[C_1]$ are acyclic subtournaments of H_r'. Let i be the source of $H_r'[C_0]$. Clearly $C_0 \subset \{i\} \cup \Gamma^+(i, H_r)$. By (i) and (ii) it follows that $|C_0| \leqslant r$. Similarly, $|C_1| \leqslant r$; and since $|C_0| + |C_1| = 2r$, $|C_0| = |C_1| = r$. By (iii) it follows that $C_0 = S_i$ or $C_0 = S_i'$.

Suppose that $C_0 = S_i'$. By (iv), $\delta^-(H_r[V(H_r) - S_i']) \geq 2$. It follows that $\delta^-(H_r'[C_1]) \geq 1$. Therefore $H_r[C_1]$ is not acyclic. Hence $C_0 = S_i$. Similarly $C_1 = S_j$ for some j. Therefore $\{C_0, C_1\} = \{S_1, S_{r+1}\}$, and Theorem 1 follows. □

Corollary 1. *$H_r - (r, 0)$, $r \geq 4$, is a uniquely colourable 2-dichromatic oriented graph. Furthermore $d_k(H_r - (j, 0)) = 3$ for $j \in \{r+2, r+3, \ldots, 2r\}$.*

Proof. $S_1 \cup \{0\}$ and S_{r+1} produce an acyclic 2-colouring of $H_r - (r, 0)$. Then $d_k(H_r - (r, 0)) = 2$. But since $d_k(H_r) = 3$, (by Theorem 2 in [7]) in any acyclic 2-colouring γ of $H_r - (r, 0)$ vertices r and 0 receive the same colour. However γ induces an acyclic 2-colouring in $H_r - 0$ whose chromatic classes are, by Theorem 1, S_1 and S_{r+1}. Then the chromatic classes of γ are $S_1 \cup \{0\}$ and S_{r+1}.

Similarly for $j \in \{r+2, r+3, \ldots, 2r\} \subset S_{r+1}$, if $H_r - (j, 0)$ were 2-dichromatic, then $S_{r+1} \cup \{0\}$ would be a chromatic class of any acyclic 2-colouring of $H_r - (j, 0)$. However $H_r[S_{r+1} \cup \{0\}] - (j, 0)$ is not acyclic. □

Remark 1. It should be pointed out that $H_3 - (u, v)$ is uniquely 2-colourable for every $(u, v) \in A(H_3)$ but $H_3 - 0$ is not.

3. Constructing uniquely colourable r-dichromatic oriented graphs, $r \geq 3$

3.1. *The function $\bar{n}$ (m_0, m_1, m_2)*

Let m_0, m_1, m_2 be three non-negative integers. The function $\bar{n}(m_0, m_1, m_2)$ was defined in [7] as the smallest integer k for which there exist three subsets J_0, J_1, J_2 of I_k such that $|J_i| = m_i$, $0 \leq i \leq 2$ and $\bigcap_{i=0,1,2} J_i = \emptyset$.

The following lemma was proved in [7].

Lemma 1. *Suppose that $m_0 \leq m_1 \leq m_2$. Then*

$$\bar{n}(m_0, m_1, m_2) = \begin{cases} m_2 & \text{if } m_0 + m_1 \leq m_2, \\ \lceil \frac{1}{2}(m_0 + m_1 + m_2) \rceil & \text{if } m_0 + m_1 \geq m_2, \end{cases}$$

We say that (m_0, m_1, m_2) is an *$\bar{n}$-upcritical triple* if $1 \leq m_i$, $i = 0, 1, 2$, and

$$\bar{n}(m_0 + 1, m_1, m_2) = \bar{n}(m_0, m_1 + 1, m_2) = \bar{n}(m_0, m_1, m_2 + 1) = \bar{n}(m_0, m_1, m_2) + 1.$$

The next result follows easily from Lemma 1.

Lemma 2. *Let $m_0 \leq m_1 \leq m_2$. Then the triple (m_0, m_1, m_2) is $\bar{n}$-upcritical if and only if $m_0 + m_1 \geq m_2$ and $m_0 + m_1 + m_2$ is even.*

Let D_0, D_1 and D_2 be three mutually disjoint digraphs. We denote by $t(D_0, D_1, D_2)$ the digraph whose vertex set is $\bigcup_{i=0}^{2} V(D_i)$ with arc set $\bigcup_{i=0}^{2} A(D_i) \cup \{(u, v) \mid u \in V(D_i),\ v \in V(D_{i+1}),\ 0 \leq i \leq 2$, where $i+1$ is taken $\bmod 3\}$. Notice that D_0, D_1 and D_2 are induced subdigraphs of $t(D_0, D_1, D_2)$.

In [7] it was proved that $d_k(t(D_0, D_1, D_2)) = \tilde{n}(m_0, m_1, m_2)$, where $d_k(D_i) = m_i$, $m_i \geq 1$, $i = 0, 1, 2$.

Lemma 3. *If (m_0, m_1, m_2) is an $\tilde{n}$-upcritical triple, then any acyclic $\tilde{n}(m_0, m_1, m_2)$-colouring of $t(D_0, D_1, D_2)$ induces an acyclic m_i-colouring of D_i, $i = 0, 1, 2$.*

Proof. Let $m = \tilde{n}(m_0, m_1, m_2)$ and let γ be an acyclic m-colouring $D = t(D_0, D_1, D_2)$. For each D_i let J_i be the set of colours used by γ in $V(D_i)$. Clearly $\bigcup_{i=0}^{2} J_i = \emptyset$ and $|J_i| \geq m_i$, $i = 0, 1, 2$. If $|J_i| \geq m_i$ for at least one $i \in \{0, 1, 2\}$, then $|\bigcup_{i=0}^{2} J_i| > \tilde{n}(m_0, m_1, m_2)$ since (m_0, m_1, m_2) is $\tilde{n}$-upcritical. This is a contradiction. □

Lemma 4. *Let (m_0, m_1, m_2) be an $\tilde{n}$-upcritical triple and γ and γ' two $\tilde{n}(m_0, m_1, m_2)$-colourings of $t(D_0, D_1, D_2)$ using the same set of colours; J_i and J_i' the sets of colours occurring in D_i in γ and γ' respectively $i = 0, 1, 2$. If $J_i = J_i'$ for two values of i, then $J_i = J_i'$, $i = 0, 1, 2$.*

Proof. We can suppose w.l.o.g. that $J_0 = J_0'$ and $J_1 = J_1'$. Since $\bigcap_{i=0}^{2} J_i = \bigcap_{i=0}^{2} J_i' = \emptyset$, we conclude that $J_0 \cap J_1 \cap (J_2 \cup J_2') = \emptyset$. It follows that $|J_2 \cup J_2'| = m_2 = |J_2| = |J_2'|$ since (m_0, m_1, m_2) is $\tilde{n}$-upcritical. Therefore $J_2 = J_2'$ □

3.2. *Construction of $D^{(l)}$*

Given a digraph D, let $D^{(l)}$ be the digraph defined by $V(D^{(l)}) = V(D) \times I_l$ and $((u, i), (v, j)) \in A(D^{(l)})$ if and only if $(u, v) \in A(D)$ and $|i - j| \leq 1$.

Remark 2. Notice that if $f: V(D) \to I_l$ is any function such that $|f(u) - f(v)| \leq 1$ for every $u, v \in V(D)$, the subdigraph of $D^{(l)}$ induced by $\{(u, f(u)) \mid u \in V(D)\}$ is isomorphic to D. In particular $D^{(l)}[V(D) \times \{i\}]$ is isomorphic to D.

Remark 3. Any acyclic r-colouring of D with chromatic classes $C_1, C_2, \ldots, C_r$ induces an acyclic r-colouring of $D^{(l)}$ with chromatic classes $C_i \times I_l$, $1 \leq i \leq r$. Thus $d_k(D) = d_k(D^{(l)})$.

Lemma 5. *Let D_i be a uniquely colourable m_i-dichromatic digraph and $D = t(D_0, D_1, D_2)$. If (m_0, m_1, m_2) is an $\tilde{n}$-upcritical triple, then in any acyclic m-colouring of $D^{(l)}$ with $m = d_k(D^{(l)})$ the sets $\{v\} \times I_l$, $v \in V(D)$ are monochromatic.*

Proof. Let $m = \tilde{n}(m_0, m_1, m_2)$, γ an acyclic m-colouring of $D^{(l)}$ and $v \in V(D)$. Suppose w.l.o.g. that $v \in V(D_0)$. It is sufficient to prove that vertices (v, i) and $(v, i+1)$, $0 \leq i < l - 1$, receive the same colour. Let H_0 be the subdigraph

$D^{(l)}[V(D) \times \{i\}]$ of $D^{(l)}$ which by Remark 2 is isomorphic to D. Using γ and H_0 we can induce an acyclic m-colouring γ_1 of D in which vertex $u \in V(D)$ receives the same colour as vertex (u, i) in γ. Similarly using γ and the subdigraph H_1 of $D^{(l)}$ induced by $((V(D) - \{v\}) \times \{i\}) \cup \{(v, i+1)\}$, $0 \leqslant i < 1-1$, we can induce a second acyclic m-colouring γ_2 of D. Since $V(H_0) - (v, i) = V(H_1) - (v, i+1)$, γ_1 and γ_2 are equal in all vertices of D except possibly in vertex v.

However, by Lemma 4, γ_1 and γ_2 induce two acyclic m_0-colourings of D_0 using the same set of colours, and since D_0 is uniquely colourable, v must receive the same colour in γ_1 and γ_2. Therefore vertices (v, i) and $(v, i+1)$ receive the same colour in γ. □

By using similar arguments the following result can be proved.

Theorem 2. *If D is a uniquely r-colourable oriented graph, then $D^{(l)}$ is also a uniquely r-colourable oriented graph.*

3.3. *Construction of $D^{(l)}(\Lambda)$*

In what follows we shall suppose that D_i is a uniquely colourable m_i-dichromatic oriented graph, $0 \leqslant i \leqslant 2$, and $D = t(D_0, D_1, D_2)$. We also assume that (m_0, m_1, m_2) is always an $\tilde{n}$-upcritical triple and $m = \tilde{n}(m_0, m_1, m_2)$. Since D_i is uniquely m_i-colourable, any acyclic m_i-colouring of D_i induces the same partition Π_i of $V(D_i)$, $i = 0, 1, 2$. Let $\Pi = \Pi_0 \cup \Pi_1 \cup \Pi_2$ and Λ any partition of $V(D)$ induced by an acyclic m-colouring of D.

For each $\alpha \in \Pi$ choose two different vertices X_α, $Y_\alpha \in \alpha$ (this is possible because in a uniquely colourable r-dichromatic oriented graph, $r \geqslant 2$, each chromatic class contains at least two elements).

If α, $\beta \in \Pi$ are not contained in the same class of Λ, let $Q(\alpha, \beta)$ be the directed square defined by:

$$V(Q(\alpha, \beta)) = \{X'_\alpha, Y'_\alpha, X''_\beta, Y''_\beta\},$$

$$A(Q(\alpha, \beta)) = \{(X'_\alpha, X''_\beta), (X''_\beta, Y'_\alpha), (Y'_\alpha, Y''_\beta), (Y''_\beta, X'_\alpha)\},$$

where $Z' = (Z, 0)$, $Z'' = (Z, l-1)$ for $Z \in V(D)$.

Let us define finally

$$D^{(l)}(\Lambda) = D^{(l)} \cup \Big(\bigcup_{\alpha,\beta} Q(\alpha, B)\Big).$$

Theorem 3. *For $l \geqslant 2$ $D^{(l)}(\Lambda)$ is a uniquely colourable m-dichromatic oriented graph.*

Proof. Clearly $d_k(D^{(l)}(\Lambda)) \geqslant m$. By Remark 3, Λ induces an acyclic m-colouring of $D^{(l)}(\Lambda)$ with chromatic classes $\lambda \times I_l$, $\lambda \in \Lambda$. It follows that $d_k(D^{(l)}(\Lambda)) = m$. Let γ be an acyclic m-colouring of $D^{(l)}(\Lambda)$.

By Lemma 5 the sets $\{v\} \times I_l$ are monochromatic in γ. Let γ' be the m-colouring of D in which vertex $v \in V(D)$ receives the same colour as vertex (v, i) in γ. Denote by Λ' the partition induced by γ' in $V(D)$ and suppose that $\Lambda' \neq \Lambda$. Since $|\Lambda'| = |\Lambda| = m$, Λ' is not a refinement of Λ and therefore we can choose $\lambda' \in \Lambda'$ which is not contained in any class of Λ. Let $\lambda_1, \lambda_2 \in \Lambda$, such that $\lambda_1 \cap \lambda'$ and $\lambda_2 \cap \lambda'$ are not empty. Take $v_i \in \lambda_i \cap \lambda'$, $i = 1, 2$ and let $\bar{v}_1$ and $\bar{v}_2$ be the classes of Π containing v_1 and v_2 respectively. Notice that $\bar{v}_i \subset \lambda_i \cap \lambda'$, $i = 1, 2$, by Lemma 3. Notice also that $\lambda_1 \neq \lambda_2$. Therefore the square $Q(\bar{v}_1, \bar{v}_2)$ is defined and is a subdigraph of $D^{(l)}(\Lambda)[\lambda' \times I_l]$ which is monochromatic in γ. This gives a contradiction. $\square$

Theorem 4. *For every $r \geq 2$ there exists an infinite family of uniquely colourable r-dichromatic oriented graphs.*

Proof. By induction over r. The case $r = 2$ follows from Theorem 1. Let us assume that the result has been proved for $2 \leq r \leq r_0$. Take three mutually disjoint uniquely colourable oriented graphs D_i, $i = 0, 1, 2$, with dichromatic numbers $m_0 = 2$, $m_2 = m_3 = r_0$. Notice that $(2, r_0, r_0)$ is a $\bar{n}$-upcritical triple and that $\bar{n}(2, r_0, r_0) = r_0 + 1$. By Theorem 3, $D^{(l)}(\Lambda)$ is a uniquely colourable $r_0 + 1$-dichromatic oriented graph for $l \geq 2$. $\square$

Some general properties of uniquely colourable r-dichromatic oriented graphs are being studied [8].

References

[1] P. Erdös, Problems and results in number theory and graph theory, Proc. Ninth Manitoba Conf. Numer. Math. and Computing (1979) 3–21.

[2] P. Erdös and V. Neumann-Lara, On the dichromatic number of graph, in preparation.

[3] Y.O. Hamidoune, On the decomposition of a minimally strongly h-connected digraph into $h + 1$ acircuitic subgraphs, Discrete Math. 31 (1980) 89–90.

[4] H. Meyniel, Extension du nombre chromatique et du nombre de stabilité, preprint.

[5] V. Neumann-Lara, The dichromatic number of a digraph, J. Combin. Theory, Ser. B 33 (1982) 265–270.

[6] V. Neumann-Lara, The generalized dichromatic number of a digraph, in: Colloq. Math. Soc. János Bolyai 37, Finite and Infinite Sets (Eger, Hungary, 1981) 601–606.

[7] V. Neumann-Lara and J. Urrutia, Vertex critical r-dichromatic tournaments, Discrete Math. 40 (1984) 83–87.

[8] V. Neumann-Lara and J. Urrutia, On uniquely colourable r-dichromatic digraphs, in preparation.